Clustering-based Mode Reduction for Markov Jump Systems

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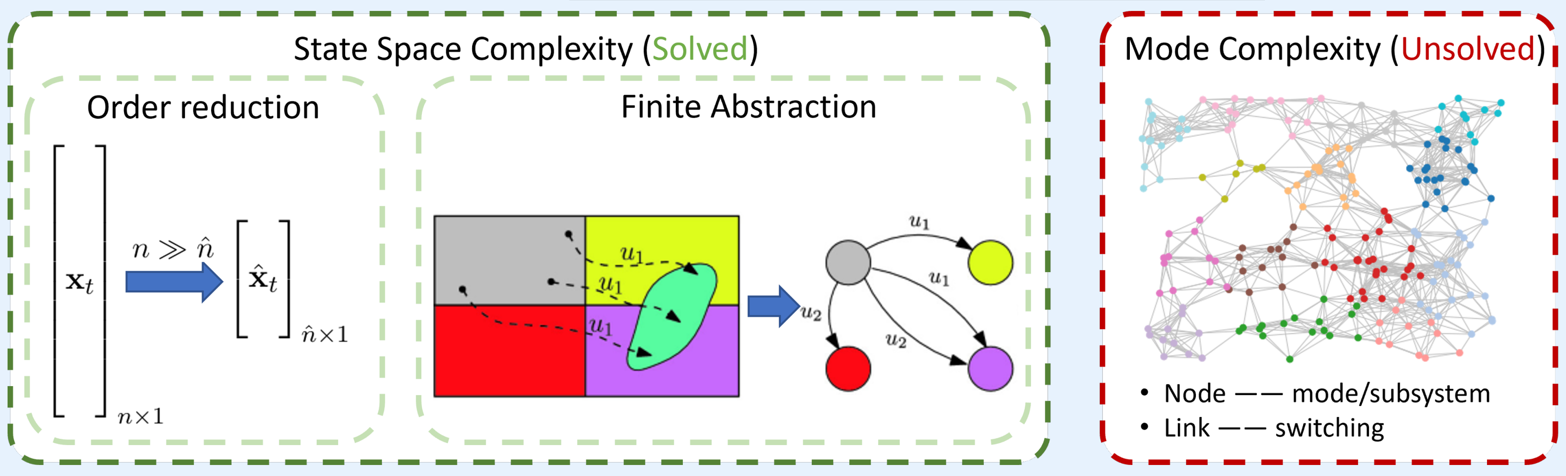
Problem Setup

➢ Time-varying systems may suffer from two types of complexities: high-dimensional states and large number of modes/subsystems.



There may exist redundancies among the modes allowing for reduced modeling.

➢ **Markov Jump Systems (MJS)**

$$\Sigma := \begin{cases} \mathbf{x}_{t+1} = \mathbf{A}_{\omega_t}\mathbf{x}_t + \mathbf{B}_{\omega_t}\mathbf{u}_t \\ \omega_0, \omega_1, \ldots \sim \text{Markov Chain}(\mathbf{T}) \end{cases}$$

- state $\mathbf{x}_t \in \mathbb{R}^n$, input $\mathbf{u}_t \in \mathbb{R}^p$
- s number of modes
  - — $\{\mathbf{A}_1, \mathbf{B}_1\}, \{\mathbf{A}_2, \mathbf{B}_2\}, \ldots, \{\mathbf{A}_s, \mathbf{B}_s\}$
- Markov matrix $\mathbf{T} \in \mathbb{R}^{s \times s}$
  - — $\text{Prob}(\omega_{t+1} = j \mid \omega_t = i) = \mathbf{T}_{ij}$

➢ **Goal**: given the s-mode Σ, construct an r-mode ($r \ll s$) MJS

$$\hat{\Sigma} := \begin{cases} \hat{\mathbf{x}}_{t+1} = \hat{\mathbf{A}}_{\hat{\omega}_t}\hat{\mathbf{x}}_t + \hat{\mathbf{B}}_{\hat{\omega}_t}\mathbf{u}_t \\ \hat{\omega}_0, \hat{\omega}_1, \ldots \sim \text{Markov Chain}(\hat{\mathbf{T}}) \end{cases}$$

with mode dynamics $\{\hat{\mathbf{A}}_i, \hat{\mathbf{B}}_i\}_{i=1}^r$ and Markov matrix $\hat{\mathbf{T}} \in \mathbb{R}^{r \times r}$.

Approach

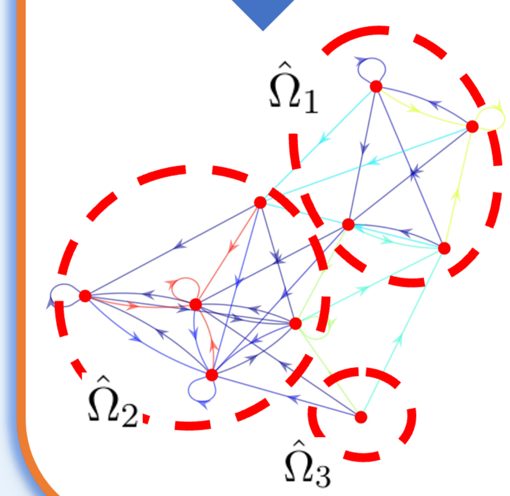
- Construct feature matrix with tuning parameter $\alpha_{\mathbf{A}}$, $\alpha_{\mathbf{B}}$, $\alpha_{\mathbf{T}}$

$$\mathbf{\Phi} := \begin{bmatrix} -\ \alpha_{\mathbf{A}} \cdot \mathbf{vec}(\mathbf{A}_1)\ - & -\ \alpha_{\mathbf{B}}\mathbf{vec}(\mathbf{B}_1)\ - & -\ \alpha_{\mathbf{T}}\mathbf{T}(1,:)\ - \\ -\ \alpha_{\mathbf{A}} \cdot \mathbf{vec}(\mathbf{A}_2)\ - & -\ \alpha_{\mathbf{B}}\mathbf{vec}(\mathbf{B}_2)\ - & -\ \alpha_{\mathbf{T}}\mathbf{T}(2,:)\ - \\ \vdots & \vdots & \vdots \\ -\ \alpha_{\mathbf{A}} \cdot \mathbf{vec}(\mathbf{A}_s)\ - & -\ \alpha_{\mathbf{B}}\mathbf{vec}(\mathbf{B}_s)\ - & -\ \alpha_{\mathbf{T}}\mathbf{T}(s,:)\ - \end{bmatrix}$$

$\mathbf{U}_r$: top-r left singular vectors

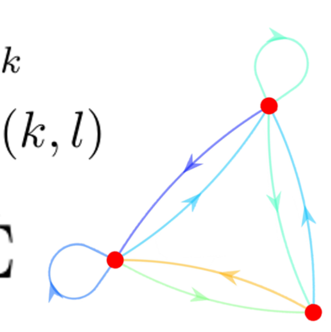
k-means:

$$\hat{\Omega}_{1:r}, \hat{\mathbf{c}}_{1:r} = \underset{\hat{\Omega}_{1:r}, \hat{\mathbf{c}}_{1:r}}{\arg\min} \sum_{k \in [r]} \sum_{i \in \hat{\Omega}_k} \|\mathbf{U}_r(i,:) - \hat{\mathbf{c}}_k\|^2$$



$$\frac{1}{|\hat{\Omega}_k|}\sum_{i \in \hat{\Omega}_k} \mathbf{A}_i \rightarrow \hat{\mathbf{A}}_k$$
$$\frac{1}{|\hat{\Omega}_k|}\sum_{i \in \hat{\Omega}_k} \mathbf{B}_i \rightarrow \hat{\mathbf{B}}_k$$
$$\frac{1}{|\hat{\Omega}_k|}\sum_{i \in \hat{\Omega}_k, j \in \hat{\Omega}_l} \mathbf{T}(i,j) \rightarrow \hat{\mathbf{T}}(k,l)$$

$\Sigma \longrightarrow \hat{\Sigma}$



Approximation Guarantees

1. Assumption and Preliminaries

Assumption (Inner-cluster Similarity). *Among the clusters $\{\hat{\Omega}_k\}_{k=1}^r$, assume for any cluster $\hat{\Omega}_k$ and any modes $i, i' \in \hat{\Omega}_k$*

$$\|\mathbf{A}_i - \mathbf{A}_{i'}\| \leq \epsilon_{\mathbf{A}}, \quad \|\mathbf{B}_i - \mathbf{B}_{i'}\| \leq \epsilon_{\mathbf{B}}, \quad \|\mathbf{T}(i,:) - \mathbf{T}(i',:)\|_1 \leq \epsilon_{\mathbf{T}}$$

for some $\epsilon_{\mathbf{A}}, \epsilon_{\mathbf{B}}, \epsilon_{\mathbf{T}} > 0$.

Definition 1 (Stability for MJS). *For Σ, define*

$$\xi(\Sigma) := \text{joint spectral radius}(\{\mathbf{A}_i\}_{i=1}^s), \quad \rho(\Sigma) := \text{spectral radius}(\boldsymbol{\mathcal{A}})$$

where $\boldsymbol{\mathcal{A}}$ is a block matrix with the ij-th block given by $\mathbf{T}_{ji} \cdot (\mathbf{A}_j \otimes \mathbf{A}_j)$. Then

$$\text{uniform stability} \iff \xi(\Sigma) < 1, \quad \text{mean-square stability} \iff \rho(\Sigma) < 1$$

2. Stability Difference

Theorem 1 (Stability Difference). *For Σ and $\hat{\Sigma}$, we have*

$$|\xi(\hat{\Sigma}) - \xi(\Sigma)| \leq \kappa\epsilon_{\mathbf{A}}$$
$$|\rho(\hat{\Sigma}) - \rho(\Sigma)| \leq \tau((2\bar{A} + \epsilon_{\mathbf{A}})\epsilon_{\mathbf{A}} + \bar{A}^2\epsilon_{\mathbf{T}})$$

where $\bar{A} = \max_i \|\mathbf{A}_i\|$, and constants κ, τ are bounded and depend on the transient responses of the MJS.

3. Trajectory Difference

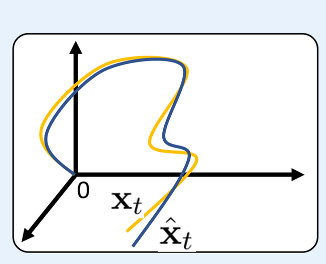
Theorem 2 (Trajectory Difference). *Suppose $\mathbf{x}_0 = \hat{\mathbf{x}}_0$, $\mathbf{u}_t \leq \bar{u}$, and $\hat{\omega}_t = k$ whenever $\omega_t \in \hat{\Omega}_k$ (mode synchrony between Σ and $\hat{\Sigma}$). Let $\rho_0 := (1 + \rho(\Sigma))/2$, $\xi_0 = (1 + \xi(\Sigma))/2$, and $\bar{B} := \max_i \|\mathbf{B}_i\|$.*

➢ *When $\xi(\Sigma) < 1$ (uniform stability), $\epsilon_{\mathbf{A}} \leq \frac{1-\xi(\Sigma)}{2\kappa}$, and $\epsilon_{\mathbf{B}} \leq \bar{B}$,*

$$\|\mathbf{x}_t - \hat{\mathbf{x}}_t\| \leq t\xi_0^{t-1}\kappa^2\|\mathbf{x}_0\|\epsilon_{\mathbf{A}} + \frac{2(1 + t\xi_0^t)\kappa^2\bar{B}\bar{u}}{1 - \xi_0}\epsilon_{\mathbf{A}} + \frac{\kappa\bar{u}}{1 - \xi_0}\epsilon_{\mathbf{B}}.$$

➢ *When $\rho(\Sigma) < 1$ (mean-square stability), $\epsilon_{\mathbf{A}} \leq \min\{\bar{A}, \frac{1-\rho(\Sigma)}{6\tau\bar{A}\|\mathbf{T}\|}\}$, and $\epsilon_{\mathbf{B}} \leq \bar{B}$,*

$$\mathbb{E}[\|\mathbf{x}_t - \hat{\mathbf{x}}_t\|] \leq 4\sqrt{ns}\tau\sqrt{t\rho_0^{t-1}\bar{A}\|\mathbf{T}\|}\|\mathbf{x}_0\|\sqrt{\epsilon_{\mathbf{A}}} + \frac{8\sqrt{ns}\bar{B}\tau\bar{u}}{(1-\rho_0)^2}\left(\sqrt{\bar{A}\|\mathbf{T}\|}\sqrt{\epsilon_{\mathbf{A}}} + \sqrt{\epsilon_{\mathbf{B}}}\right).$$



4. LQR Controller Suboptimality

Definition 2 (MJS-LQR). *Given Σ, positive definite cost matrices $\mathbf{Q}$ and $\mathbf{R}$, define the following quadratic cost w.r.t. controllers $\mathcal{K} = \{\mathbf{K}_i\}_{i=1}^s$,*

$$J_\Sigma(\mathcal{K}) := \limsup_{T \to \infty} \mathbb{E}\left[\frac{1}{T}\sum_{t=0}^{T} \mathbf{x}_t^\intercal\mathbf{Q}\mathbf{x}_t + \mathbf{u}_t^\intercal\mathbf{R}\mathbf{u}_t\right] \text{ s.t. } \mathbf{x}_t \sim \Sigma \text{ with } \mathbf{u}_t = \mathbf{K}_{\omega_t}\mathbf{x}_t$$

Let $J^\star = \min_{\mathcal{K}} J_\Sigma(\mathcal{K})$ denote the optimal cost.

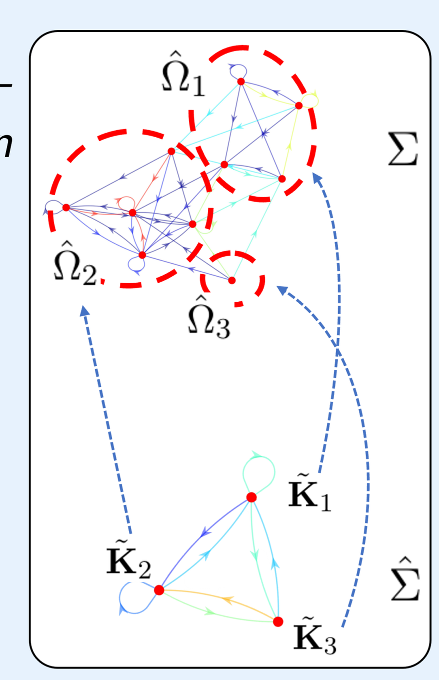
Theorem 3 (LQR Suboptimality). *Suppose Σ is mean-square stabilizable with additive noise $\mathcal{N}(0, \sigma_{\mathbf{w}}^2\mathbf{I})$. Design controllers using $\hat{\Sigma}$ and deploy them to Σ:*

$$\{\hat{\mathbf{K}}_k\}_{k=1}^r := \arg\min_{\{\hat{\mathbf{K}}_k\}_{k=1}^r} J_{\hat{\Sigma}}(\{\hat{\mathbf{K}}_k\}_{k=1}^r),$$
$$\hat{\mathcal{K}} := \{\hat{\mathbf{K}}_i\}_{i=1}^s \text{ s.t. } \hat{\mathbf{K}}_i := \hat{\mathbf{K}}_k \text{ if } i \in \hat{\Omega}_k,$$

and let $J = J_\Sigma(\hat{\mathcal{K}})$. Then for small enough $\epsilon_{\mathbf{A}}, \epsilon_{\mathbf{B}}$, and $\epsilon_{\mathbf{T}}$,

$$J - J^\star \leq C\sigma_{\mathbf{w}}^2(ns)^{1.5}(\epsilon_{\mathbf{A}} + \epsilon_{\mathbf{B}} + \epsilon_{\mathbf{T}})^2$$

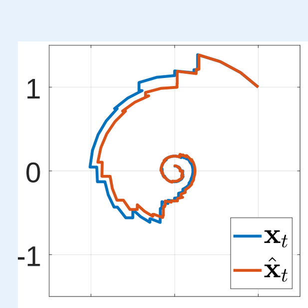
where C is some problem dependent constant.



Experiments

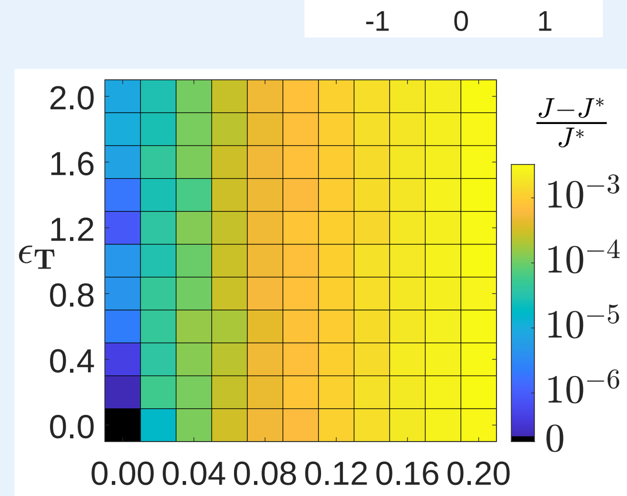
➢ Trajectory Difference

- Number of modes: $s = 30$ (original), $r = 3$ (after reduction)
- For $i = 1, \ldots, 10$, $\mathbf{A}_i = [\cos(\theta_i), -\sin(\theta_i); \sin(\theta_i), \cos(\theta_i)]$ with $\theta_i \sim \frac{\pi}{16}\text{unif}(0.9, 1.1)$;
- For $i = 11, \ldots, 20$, $\mathbf{A}_i = [a_i, 0; 0, 1]$ with $a_i \sim \text{unif}(0.9, 1)$
- For $i = 21, \ldots, 30$, $\mathbf{A}_i = [1, 0; 0, a_i]$ with $a_i \sim \text{unif}(0.9, 1)$
- $\mathbf{B}_i = 0$, $\mathbf{T} = (\mathbf{1}_s\mathbf{1}_s^\intercal/s + \mathbf{I}_s)/2$, $\mathbf{x}_0 = [1, 1]^\intercal$



➢ LQR Controller Suboptimality

- Number of modes: $s = 100$ (original), $r = 4$ (after reduction)
- $n = 10, p = 5, \sigma_{\mathbf{w}} = 0.1, \mathbf{x}_0 = \mathbf{1}$
- Randomly generated dynamics Σ and cost matrices $\mathbf{Q}$ and $\mathbf{R}$.
- Controllers are solved via Riccati iterations with tolerance 10^{-12}.
- The plot shows the normalized suboptimality $\frac{J-J^\star}{J^\star}$ vs. $\epsilon_{\mathbf{T}}, \epsilon_{\mathbf{A}} = \epsilon_{\mathbf{B}}$ averaged over 100 runs of experiments.



Summary

➢ For more details and results, see the complete version (https://arxiv.org/abs/2205.02697).

- clustering performance guarantees
- weaker assumptions
- stronger trajectory difference guarantees w/o mode synchrony

➢ Future work

- Extend to the case when the output is only partially observed, i.e. $\mathbf{y}_t = C_{\omega_t}\mathbf{x}_t$ for some measurement matrices $\{\mathbf{C}_i\}_{i=1}^s$.
- Consider the case when there can be infinite number of modes.